Categories as type classes in the Scala Algebra System

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* The Scala Algebra System

* Categorical view of computer algebra

* Bounded polymorphism

* Type classes

* polynomial arithmetic over various base rings (integer, rational, complex, modular)
* rational and algebraic functions
* ring modules/algebras
* polynomial GCD
* Gröbner bases

Trade-off

* type-safe
* generic
* mathematical syntax
* efficient ?

* 2003 : jscl-meditor

* 2004 : jscl was chosen as symbolic engine of GeoGebra

* 2007 : encounter with JAS and parametric polymorphism (Java 2.5)

* 2008 : port to Scala -> ScAS

* 2012 : ScAS 2.0 (type classes)

* 2013 : scripting support in Scala 2.11

1. arithmetic operations between elements are restricted based on their **domain**

2. domains must support abstraction by means of **categories**

3. categories must support multiple inheritance

4. (optional) categories may support default definition of operations

Categorical view of computer algebra

* Axiom [Davenport:1992]

* Gauss [Gruntz:1994]

* JAS [Kredel:2006]

* DoCon [Mechveliani:2001]

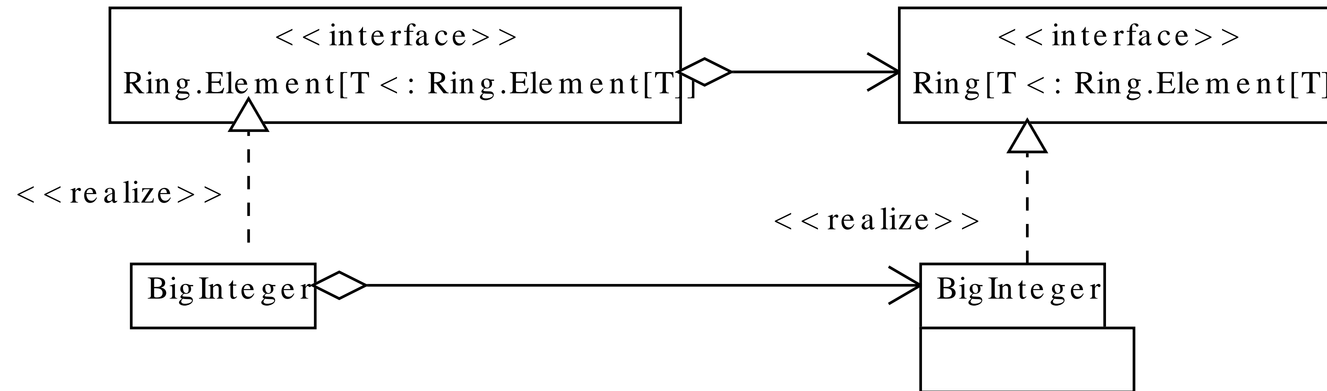
* Mathemagix [VanDerHoeven:2002]

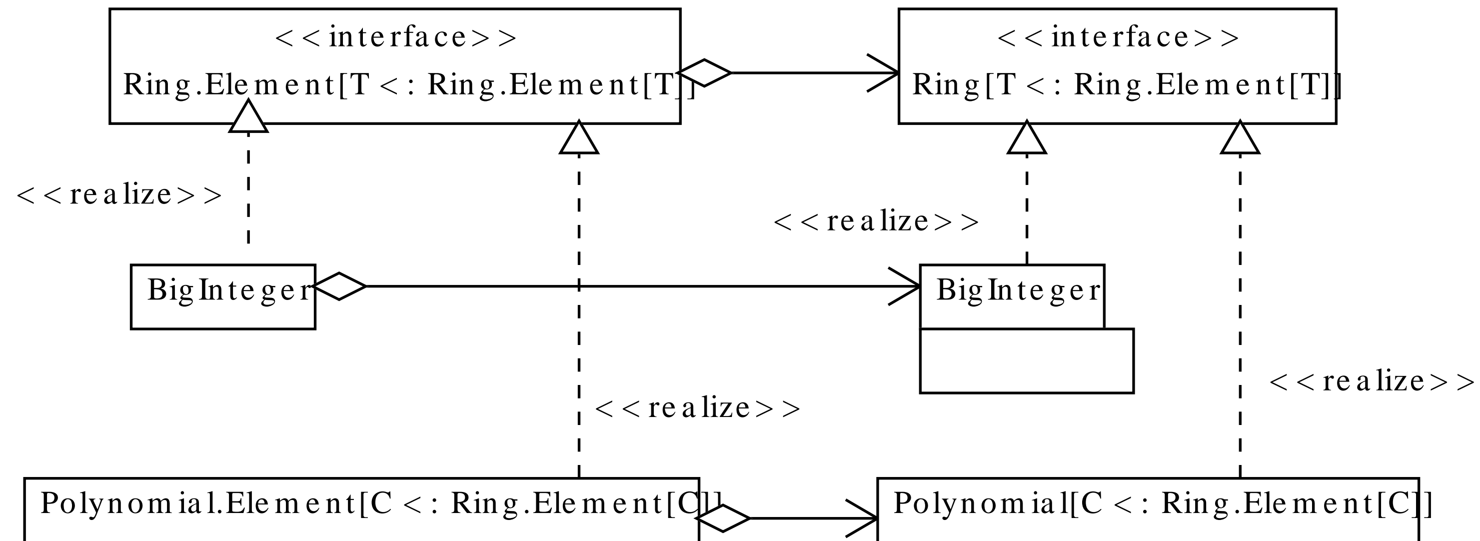
```
trait Ring[T <: Ring.Element[T]] {
  def zero: T
  ...
}
object Ring {
  trait Element[T <: Element[T]] {
    val factory: Ring[T]
    def +(that: T): T
    def unary_-: T
    ...
  }
}
```

```
object BigInteger extends Ring[BigInteger] {
  def apply(i: Int): BigInteger =
      apply(java.math.BigInteger.valueOf(i))
  def apply(value: java.math.BigInteger) =
      new BigInteger(value)
  def zero = apply(0)
}
class BigInteger(val value: java.math.BigInteger)
    extends Ring.Element[BigInteger] {
  val factory = BigInteger
  def +(that: BigInteger) = factory(
      this.value.add(that.value))
  def unary_- = factory(value.negate())
  override def toString = value.toString
}
BigInteger(1) + BigInteger(1)
// 2
```

```
class Polynomial[C <: Ring.Element[C]](val ring: Ring[C],
    val variable: String) extends Ring[
    Polynomial.Element[C]] {
  def generator = new Polynomial.Element(...)(this)
}
object Polynomial {
  def apply[C <: Ring.Element[C]](ring: Ring[C],
      variable: String) = new Polynomial(ring, variable)
  class Element[C <: Ring.Element[C]](val value: Repr[C])(
      val factory: Polynomial[C]) extends Ring.Element[
      Element[C]]
}
val r = Polynomial(BigInteger, "x")
val x = r.generator
x + x
// 2*x
```

Basic Types : BigInteger




<<interface>>
Ring.Element[T <: Ring.Element[T]]
<<interface>>
Ring[T <: Ring.Element[T]]
<<realize>>
<<realize>>
BigInteger
BigInteger
<<realize>>
<<realize>>
Polynomial.Element[C <: Ring.Element[C]]
Polynomial[C <: Ring.Element[C]]

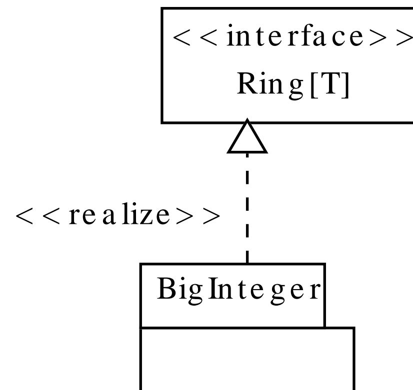
```
trait Ring[T] { outer =>
  def zero: T
  def plus(x: T, y: T): T
  implicit def mkOps(value: T): Ring.Ops[T] = new Ring.Ops[T] {
    val lhs = value
    val factory = outer
  }
}
object Ring {
  trait ExtraImplicits {
    implicit def infixRingOps[T: Ring](lhs: T) =
        implicitly[Ring[T]].mkOps(lhs)
  }
  trait Ops[T] {
    val lhs: T
    val factory: Ring[T]
    def +(rhs: T) = factory.plus(lhs, rhs)
  }
}
```

```
type BigInteger = java.math.BigInteger
object BigInteger extends Ring[BigInteger] {
  def apply(i: Int) = java.math.BigInteger.valueOf(i)
  def zero = apply(0)
  def plus(x: BigInteger, y: BigInteger) = x.add(y)
}
trait ExtraImplicits {
  implicit val ZZ = BigInteger
}
object Implicits extends ExtraImplicits
    with Ring.ExtraImplicits

import Implicits.{ZZ, infixRingOps}
BigInteger(1) + BigInteger(1)
// 2
```

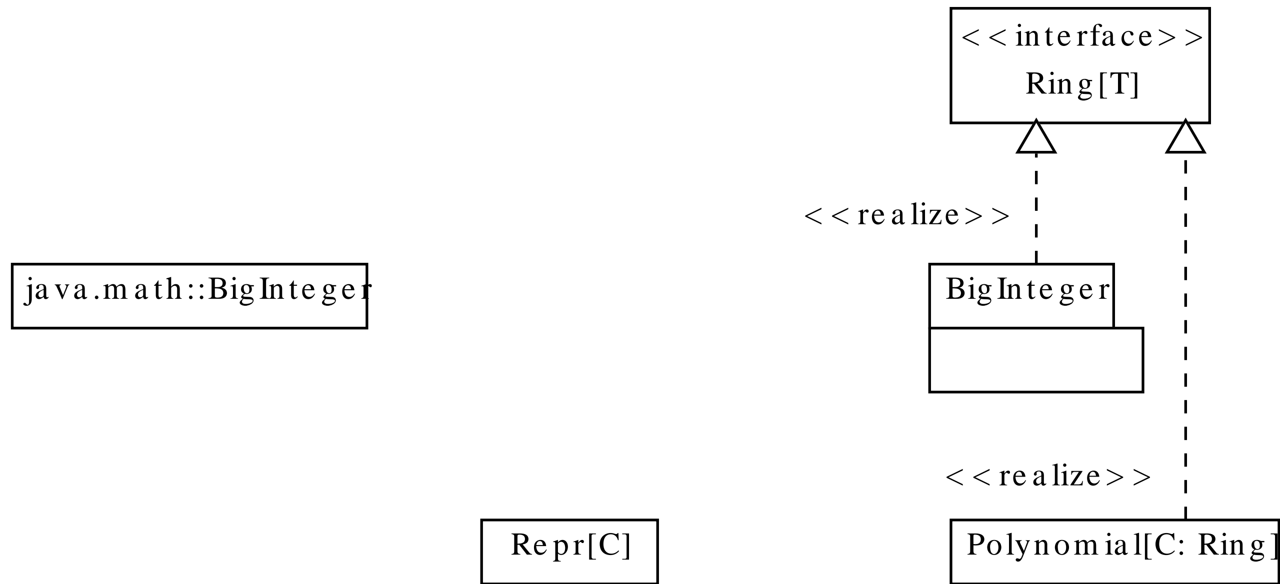
```
class Polynomial[C: Ring](val variable: String)
    extends Ring[Repr[C]] {
  val ring = implicitly[Ring[C]]
  def generator: Repr[C] = ...
}
object Polynomial {
  def apply[C: Ring](variable: String) =
      new Polynomial(variable)
}

import Implicits.{ZZ, infixRingOps}
implicit val r = Polynomial[BigInteger]("x")
val x = r.generator
x + x
// 2*x
```

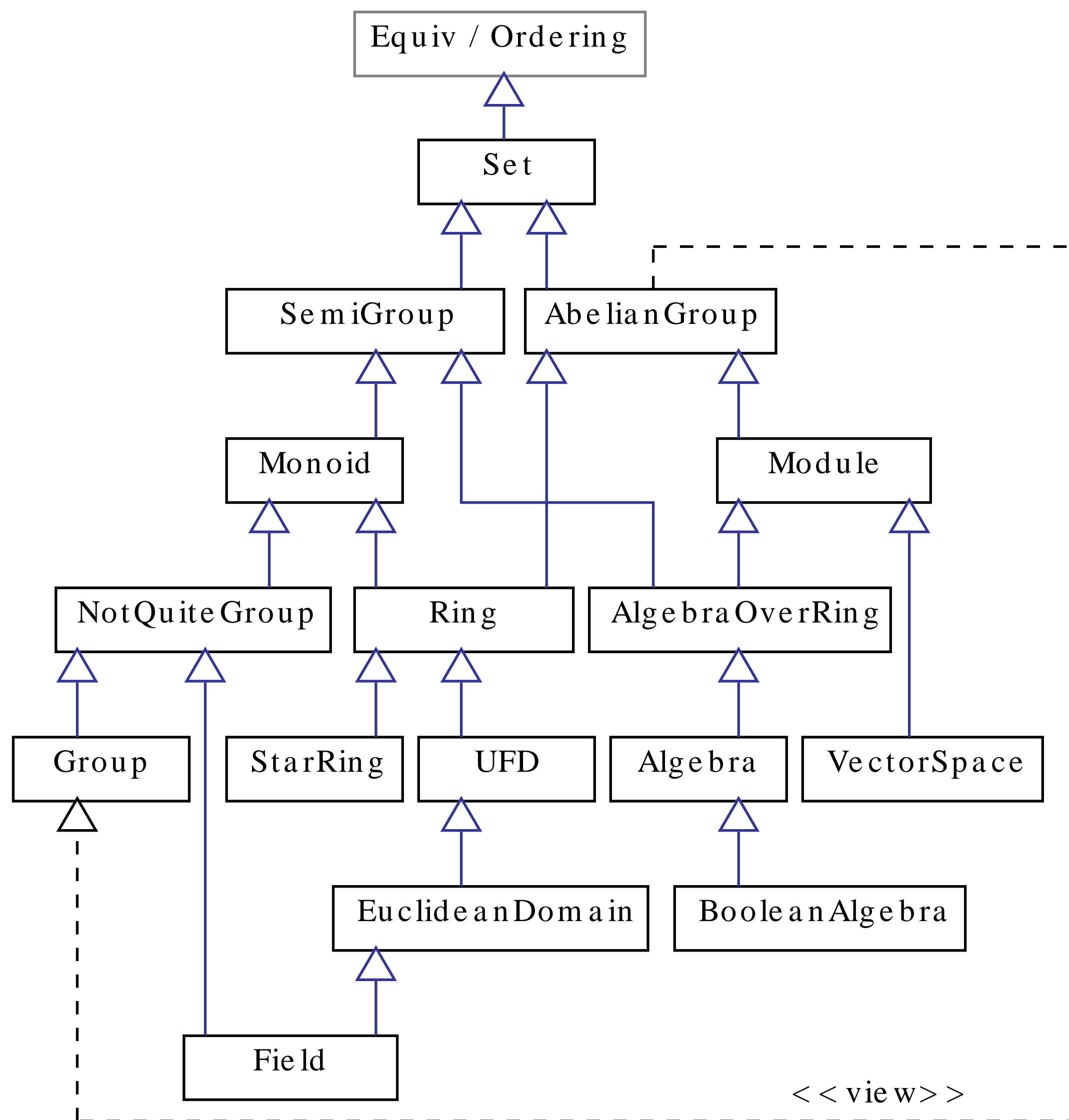
Basic Types : BigInteger

java.math::BigInteger



Basic Types : BigInteger, Polynomial




Equiv / Ordering
Set
SemiGroup
AbelianGroup
Monoid
Module
NotQuiteGroup
Ring
AlgebraOverRing
Group
StarRing
UFD
Algebra
VectorSpace
EuclideanDomain
BooleanAlgebra
Field
<<view>>

Type classes : pros

* avoid the dependent type problem : type system makes no distinction between types of elements from e.g. ModInteger(2) and ModInteger(3) but only one implicit value is allowed for the associated domain

* reuse of existing types : allows unboxed primitive types => generic numeric-symbolic implementations

* makes domain subclassing easier :

```
SolvablePolynomial <: Polynomial
RealAlgebraicNumber <: AlgebraicNumber
PolynomialWithSubresGCD <: Polynomial
```

* does not play well with coercions

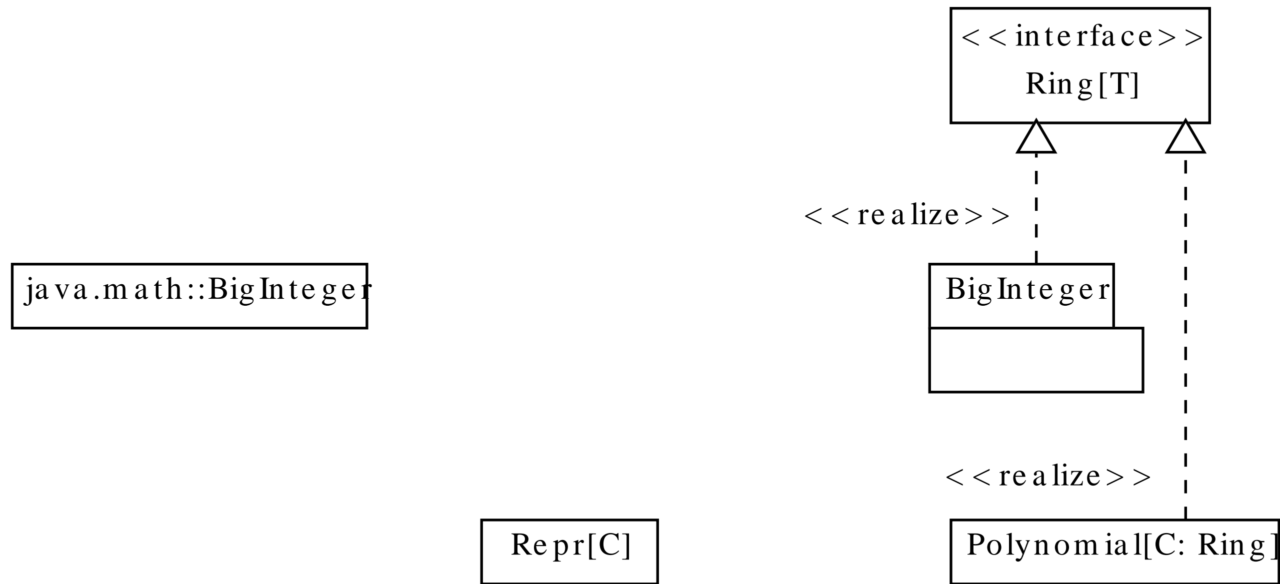
```
BigInteger(1) + 1 => BigInteger(1) + BigInteger(1)
x + 1             => x + r(1)
```

* Need an idea to solve this

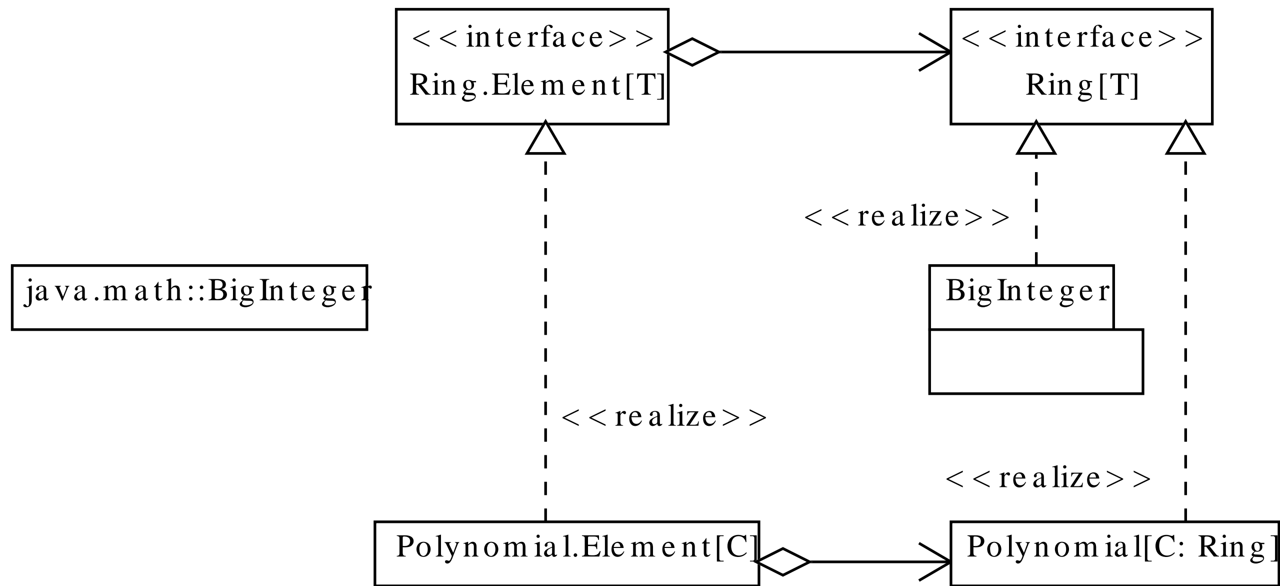
```
object Ring {
  trait ExtraImplicits {
    implicit def infixRingOps[T: Ring](lhs: T) =
        implicitly[Ring[T]].mkOps(lhs)
  }
  trait Ops[T] {
    val lhs: T
    val factory: Ring[T]
    def +(rhs: T) = factory.plus(lhs, rhs)
  }




}
```

```
object Ring {
  trait ExtraImplicits {
    implicit def infixRingOps[T: Ring](lhs: T) =
        implicitly[Ring[T]].mkOps(lhs)
  }
  trait Ops[T] {
    val lhs: T
    val factory: Ring[T]
    def +(rhs: T) = factory.plus(lhs, rhs)
  }
  trait Element[T <: Element[T]] extends Ops[T] {
      this: T =>
    val lhs = this
  }
}
```

Hybrid abstraction scheme



Hybrid abstraction scheme



Type classes in computer algebra

* introduced in Haskell [Oliveira:2010]

* first mentionned as possible abstractions for computer algebra structures [Weber:1993] [Santas:1995]

* DoCon : actually uses them

* Scala : a code example in the language documentation explicitly involves abstract algebraic constructs

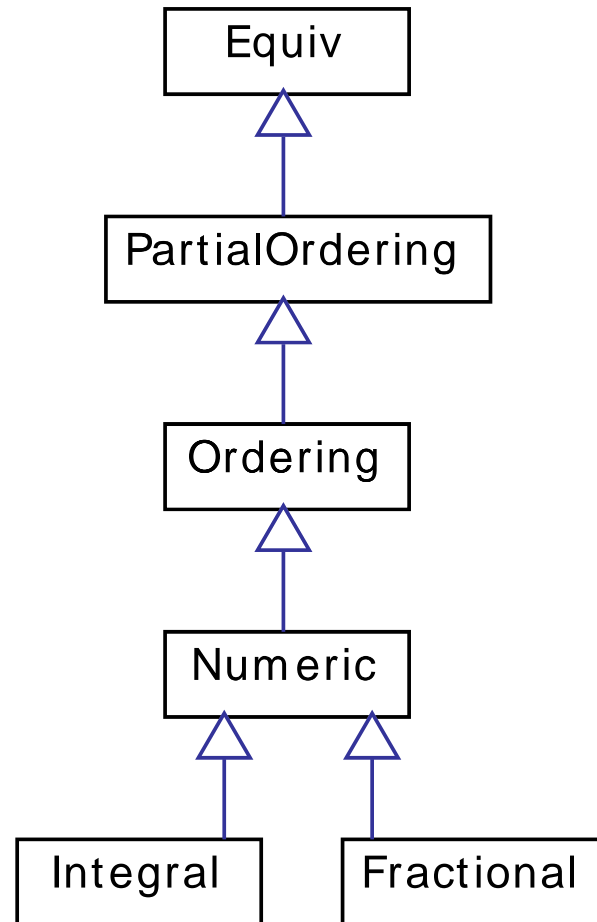
* Mathemagix uses a concept of categories that is completely equivalent to type classes

```
class CommutativeRing a => GCDRing a where
    gcd : a -> a -> a
  canAssoc : a -> a

instance GCDRing Integer where
    canAssoc n = if n < 0 then -n else n
  gcd n 0 = n
  gcd n m = gcd m (mod n m)

data Fraction a = a :/ a ...
instance GCDRing a =>
    AdditiveSemigroup (Fraction a) where
  (x :/ y) + (x' :/ y') =
      ... usual way to sum fractions

instance GCDRing a => Field (Fraction a) where ...
```

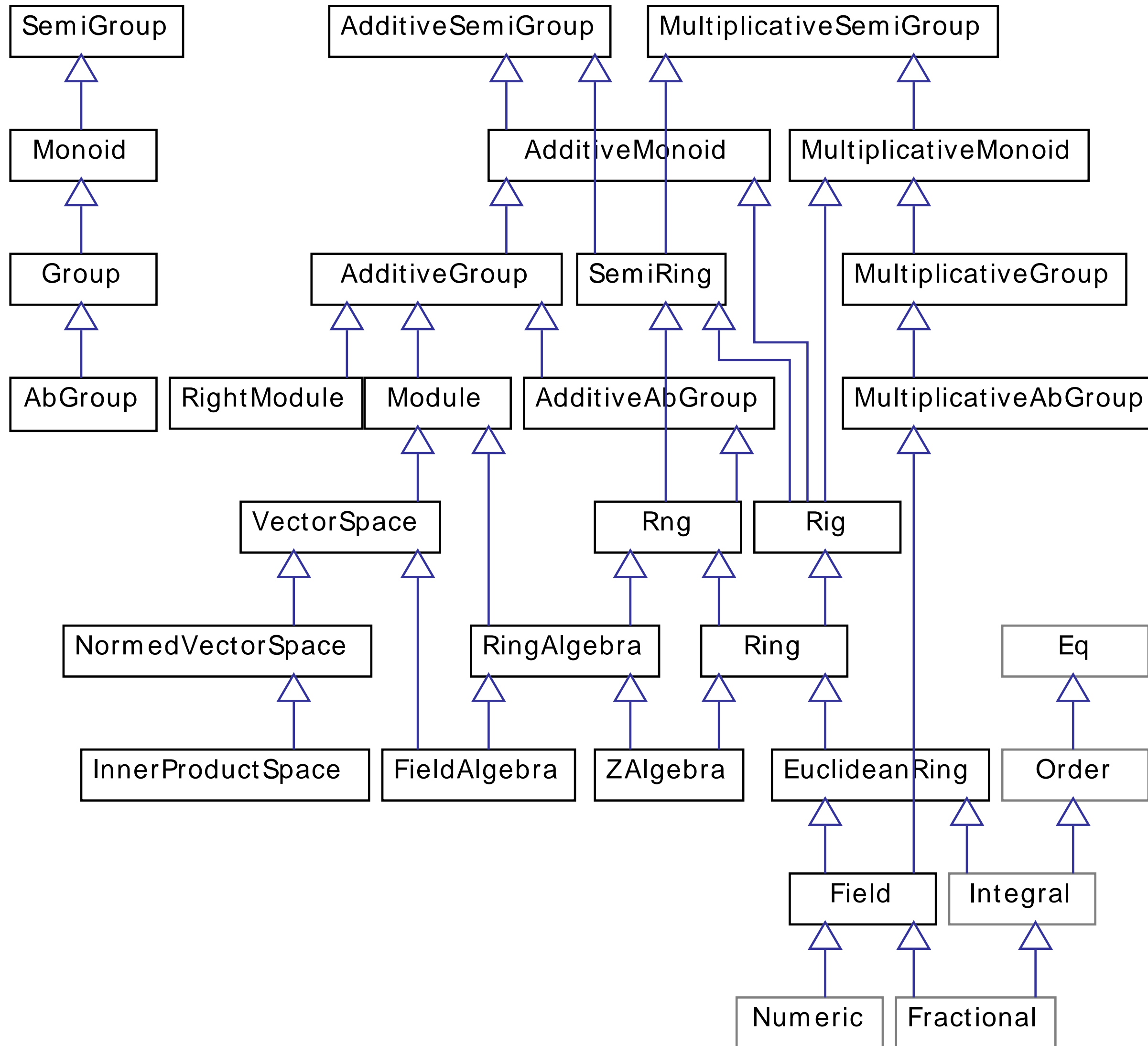
```
abstract class SemiGroup[A] {
  def add(x: A, y: A): A
}
abstract class Monoid[A] extends SemiGroup[A] {
  def unit: A
}
object ImplicitTest extends App {
  implicit object StringMonoid extends Monoid[String] {
    def add(x: String, y: String): String = x concat y
    def unit: String = ""
  }
  implicit object IntMonoid extends Monoid[Int] {
    def add(x: Int, y: Int): Int = x + y
    def unit: Int = 0
  }
  def sum[A](xs: List[A])(implicit m: Monoid[A]): A =
    if (xs.isEmpty) m.unit
    else m.add(xs.head, sum(xs.tail))

  println(sum(List(1, 2, 3))) // 6
  println(sum(List("a", "b", "c"))) // abc
}
```

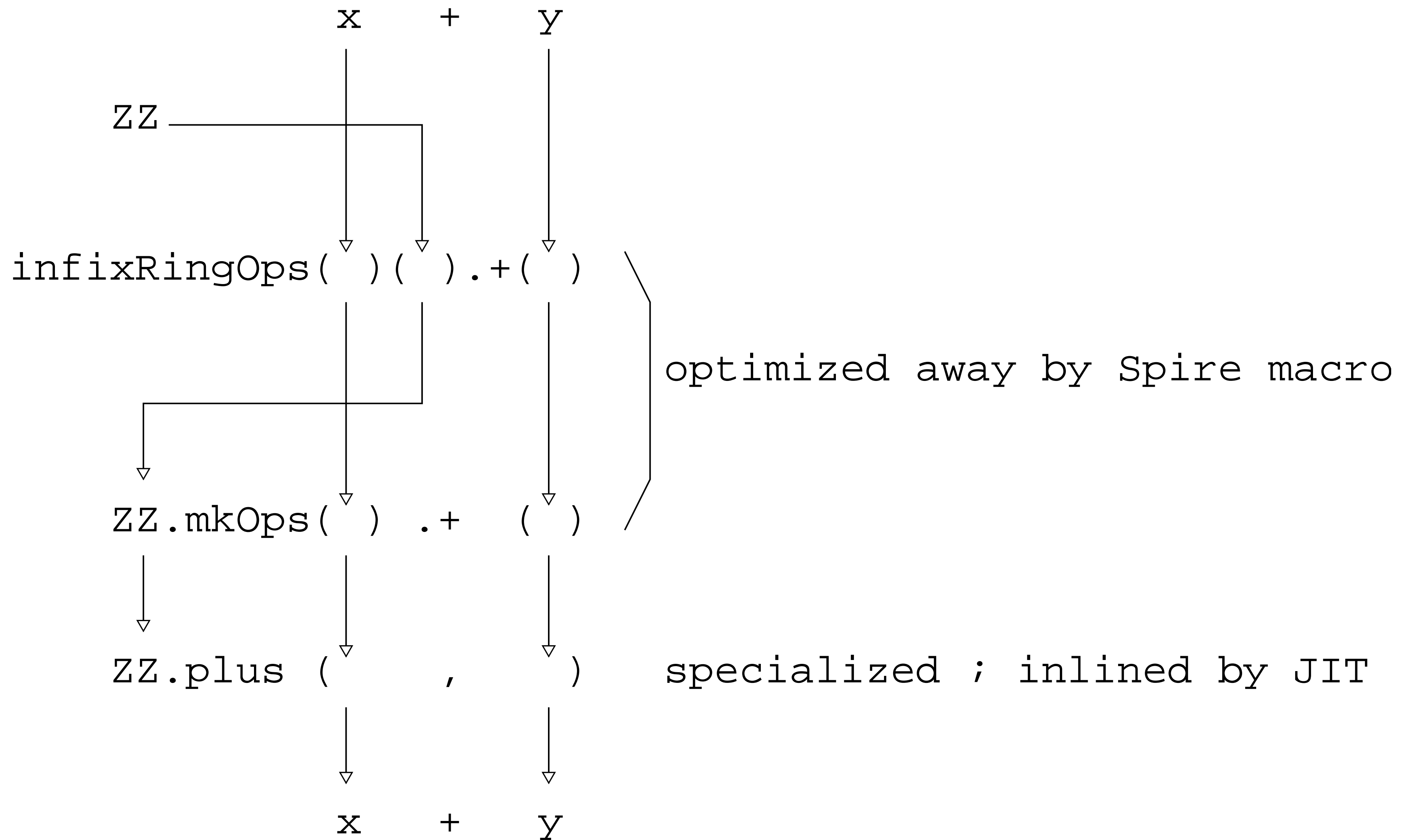
http://docs.scala-lang.org/tutorials/tour/implicit-parameters.html

Categories in the Scala standard library



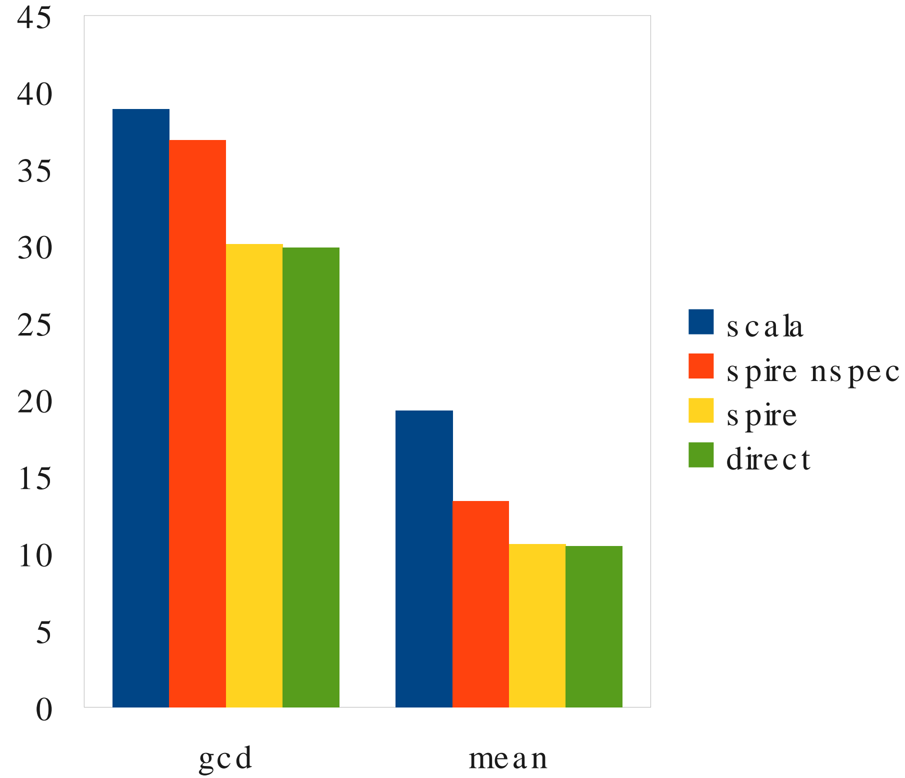
Categories in Spire




x + y
ZZ
infixRingOps( )( ).+( )
optimized away by Spire macro
ZZ.mkOps( ) .+ ( )
ZZ.plus (,)
specialized ; inlined by JIT
x + y

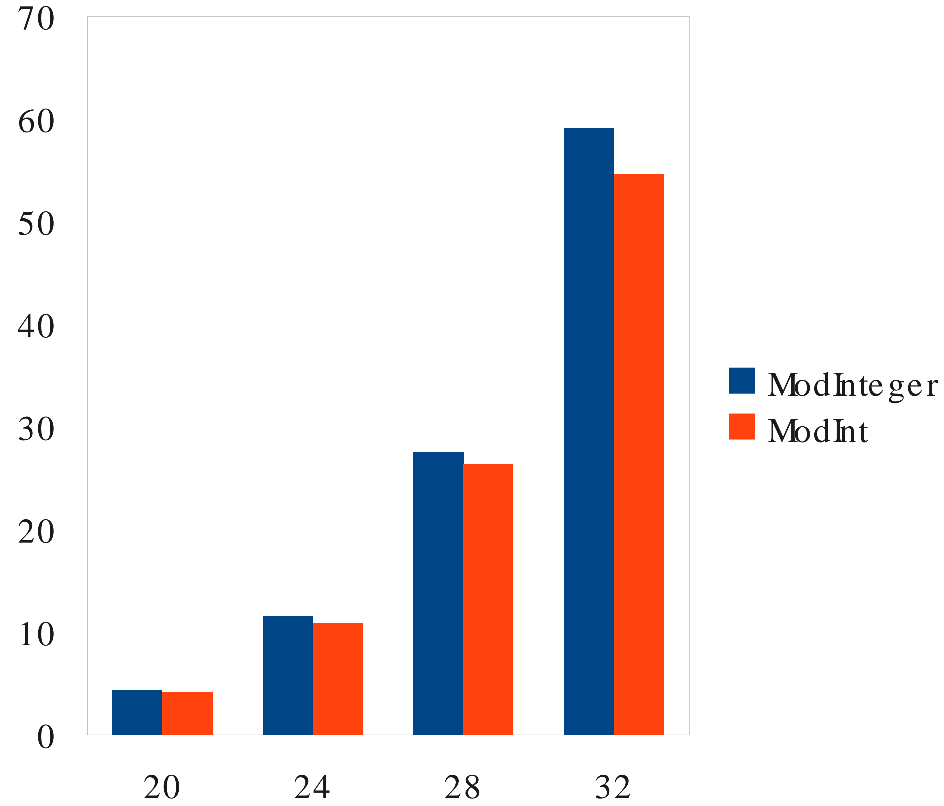
Spire is Fast

	scala	spire nspec	spire	direct
gcd	38.9	36.9	30.1	29.9
mean	19.3	13.4	10.6	10.5



http://typelevel.org/blog/2013/07/07/generic-numeric-programming.html

n	ModInteger	ModInt
20	4.4	4.2
24	11.6	10.9
28	27.6	26.4
32	59.1	54.6



Thank you !

http://github.com/rjolly/scas